SOLUTIONS TO SELECTED PRACTICE PROBLEMS FOR EXAM 2

1. Find the absolute minimum and maximum values of the function $f(x, y) = \frac{1}{2}x^2 + y^2$ on the elliptic disk $D : 0 \leq \frac{1}{2}x^2 + y^2 \leq 1$ by finding critical points on the interior of D and using a Lagrange multiplier on the boundary of D.

Solution. On the interior of D we solve

$$f_x = x = 0$$
$$f_y = 2y = 0$$

so that (0,0) is the only critical point in the interior of D. On the boundary of D, the resulting constraint is the same as the function, and since the largest value of the constraint is 1, we have that the maximum value of $f(x, y)$ on the boundary of D equals 1. Thus, over the region D, the maximum value is 1 and the minimum value is 0. Note that we can still get the same answer using a Lagrange multiplier. If we set $g(x, y) = \frac{1}{2}x^2 + y^2 = 1$, from $\nabla f = \lambda \nabla g$, we have

$$x = \lambda x$$
$$2y = 2y.$$

Notice that when $\lambda = 1$, all points satisfying the constraint equation satisfy these equations, so that every point on the boundary of D is a critical point, and each of these points yields the maximum value of 1 for $f(x, y)$ along the boundary of D.

2. Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2 - x + y$ subject to the constraint $x^2 + y^2 + z^2 = 1$. Use this to find the absolute maximum and minimum values of $f(x, y, z)$ on the solid $0 \leq x^2 + y^2 + z^2 \leq 1$. Hint: The critical points on the interior of D satisfy $f_x = f_y = f_z = 0$.

Solution. For the critical points in the interior of the sphere we have

$$f_x = 2x - 1 = 0$$
$$f_y = 2y + 1 = 0$$
$$f_z = 2z = 0$$

from which it follows that $(-\frac{1}{2}, -\frac{1}{2}, 0)$ is a critical point. Using $g(x, y, z) = x^2 + y^2 + z^2 = 1$ for the constraint equation, upon setting $\nabla f = \lambda nbg$, we have

$$2x - 1 = \lambda 2x$$
$$2y + 2 = \lambda 2y$$
$$2z = \lambda 2z.$$

If $z \neq 0$, then third equation implies $\lambda = 1$, but this leads to a contradiction by setting $\lambda = 1$ in the first equation. Thus, we must have $z = 0$. Furthermore, if we multiply the first equation by y and the second equation by x and subtract, we get $-y - x = 0$, so $y = -x$. Using this in the constraint equation we have $x^2 + (-x)^2 + 0^2 = 1$, so that $2x^2 = 1$, so $x = \pm\frac{\sqrt{2}}{2}$. Thus critical points on the boundary are: $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$. From the equations

$$f(\frac{1}{2}, -\frac{1}{2}, 0) = -\frac{1}{2}$$
$$f(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0) = 1 - \sqrt{2}$$
$$f(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) = 1 + \sqrt{2}$$

we see that the absolute maximum value of $f(x, y, z)$ over the solid sphere is $1+\sqrt{2}$ and the absolute minimum value is $1-\sqrt{2}$. These are also the maximum and minimum values of $f(x, y, z)$ subject to the given constraint.

3. Find the extreme values of $f(x, y, z) = x + y + z$ subject to the constraints $x^2 + y^2 = 2$ and $x + z - 1 = 0$.

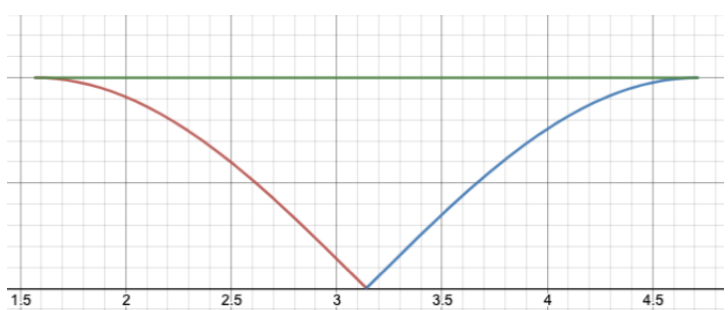
Solution. Writing the constraints as $g_1(x, y, z) = x^2 + y^2 - 2 = 0$ and $g_2(x, y, z) = x + z - 1 = 0$ and setting $\nabla f = \lambda_2 \nabla g_1 + \lambda_2 \nabla g_2$ we have

$$\begin{aligned} 1 &= \lambda_1 2x + \lambda_2 \\ 1 &= \lambda_1 2y \\ 1 &= \lambda_2. \end{aligned}$$

Setting $\lambda_2 = 1$ in the first equation gives $2\lambda_1 x = 0$. The second equation preclude $\lambda_1 = 0$, so $x = 0$. Using this in the second constraint gives $z = 1$. Putting $x = 0$ in the first constraint gives $y^2 = 2$, so $y = \pm\sqrt{2}$. Thus, the critical points are: $(0, \sqrt{2}, 1)$ and $(0, -\sqrt{2}, 1)$. Evaluating $f(x, y, z)$ at these critical points yields a maximum value of $1+\sqrt{2}$ and a minimum value of $1-\sqrt{2}$.

5. OS Chapter 5: # 105: Find the volume under the graph of $z = x^3$ above the region D in the plane bounded by $x = \sin(y), x = -\sin(y), x = 1$, with $\frac{\pi}{2} \leq y \leq \frac{3\pi}{2}$.

Solution. Without loss of generality, we interchange the roles of x and y, so that we want $\int\int_D y^3 \, dA$, with D pictured below.



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where the brown line is that portion of $y = \sin(x)$ with $\frac{\pi}{2} \leq x \leq \pi$ and the blue line is that portion of $y = -\sin(x)$, with $\leq x \leq \frac{3\pi}{2}$. The green line is the corresponding part of $y = 1$. Thus, the volume in question is:

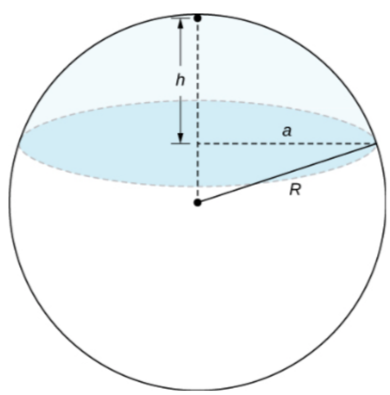
$$\int_{\frac{\pi}{2}}^{\pi} \int_{sin(x)}^{2} y^3 \, dy \, dx + \int_{\pi}^{\frac{3\pi}{2}} \int_{-sin(x)}^{1} y^3 \, dy \, dx.$$

To calculate these integrals, we will need the formula $\sin^4(x) = \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$, which can be derived from the double angle formulas for sine and cosine. For the first of the two integrals we have

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi}\int_{sin(x)}^{2} y^3\ dy\ dx &= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} y^4\Big|_{y=\sin(x)}^{y=1}\ dx \\
&= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} 1 - \sin^4(x)\ dx \\
&= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} 1 - (\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x))\ dx \\
&= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} \frac{5}{8} + \frac{1}{2}\cos(2x) - \frac{1}{8}\cos(4x)\ dx \\
&= \frac{1}{4}(\frac{5}{8}x + \frac{1}{4}\sin(2x) - \frac{1}{32}\sin(4x))\Big|_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{4}\{(\frac{5}{8}\pi + 0 - 0) - (\frac{5}{8}\cdot\frac{\pi}{2} + 0 - 0)\} \\
&= \frac{5\pi}{64}.
\end{aligned}$$

Either by symmetry or essentially the same calculation, the second integral also equals $\frac{5\pi}{64}$. Thus the required volume is $\frac{5\pi}{64} + \frac{5\pi}{64} = \frac{5\pi}{32}$.

5. OS Chapter 5: #178. (a) Show that the volume of the spherical cap below equals $\frac{\pi h}{6}(3a^2 + h^2)$.



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Solution. The first thing to notice is that the domain of integration will be $D : 0 \leq x^2 + y^2 \leq a^2$. However, the double integral of the top half of the sphere over D gives all of the volume above D and under the sphere, which is more than the cap. The excess amount is the cylinder of radius a and height $R - h$ that the spherical cap sits on. Thus, the volume we seek is

$$\int\int_D \sqrt{R^2 - x^2 - y^2}\ dA - \pi a^2(R - h).$$

Calculating the double integral using polar coordinates, we have

$$\begin{aligned}
\int\int_D \sqrt{R^2 - x^2 - y^2}\ dA &= \int_0^{2\pi} \sqrt{R^2 - r^2}\ r\ dr\ d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}(R^2 - r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=a}\ d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}(R^2 - a^2)^{\frac{3}{2}} + \frac{1}{3}R^3\ d\theta \\
&= \frac{2\pi}{3}\{-(R^2 - a^2)^{\frac{3}{2}} + R^3\}.
\end{aligned}$$

Thus, the volume of the spherical cap is

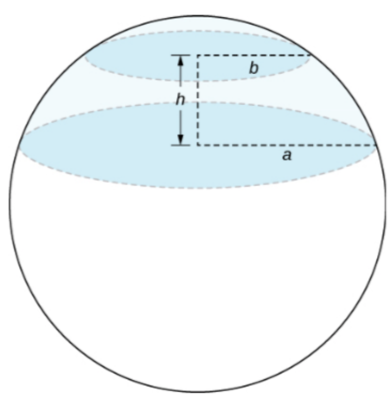
$$\frac{2\pi}{3}\{-(R^2-a^2)^{\frac{3}{2}}+R^3\}-\pi a^2(R-h).$$

Now we have to do a bit of algebra to put this expression into the required form. From the picture above we see that $R^2=(R-h)^2+a^2$. This tells us that $R^2-a^2=(R-h)^2$, from which it follows that $(R^2-a^2)^{\frac{3}{2}}=(R-h)^3$. Expanding $R^2-a^2=(R-h)^2=R^2-2Rh+h^2$ also gives that $2Rh=h^2+a^2$. Solving for R we get $R=\frac{h^2+a^2}{2h}$ and $R-h=\frac{a^2-h^2}{2h}$. We will use these equations below.

We now have that the volume we seek is

$$\begin{aligned}
\frac{2\pi}{3}\{-(R^2-a^2)^{\frac{3}{2}}+R^3\}-\pi a^2(R-h) &= \frac{2\pi}{3}\{-(R-h)^3+R^3\}-\pi a^2(R-h)\\
&= \frac{2\pi}{3}\{-R^3+3R^2h-3Rh^2+h^3+R^3\}-\pi a^2(R-h)\\
&= \frac{2\pi}{3}\{3Rh(R-h)+h^3\}-\pi a^2(R-h)\\
&= 2\pi Rh(R-h)+\frac{2\pi}{3}h^3-\pi a^2(R-h)\\
&= (R-h)\{2\pi Rh-\pi a^2\}+\frac{2\pi}{3}h^3\\
&= \frac{a^2-h^2}{2h}\{\pi\cdot(h^2+a^2)-\pi a^2\}+\frac{2\pi}{3}h^3\\
&= \frac{a^2-h^2}{2h}\pi\ h^2+\frac{2\pi}{3}h^3\\
&= \frac{\pi a^2h}{2}-\frac{\pi}{2}h^3+\frac{2\pi}{3}h^3\\
&= \frac{\pi a^2h}{2}+\frac{\pi h^3}{6}\\
&= \frac{\pi h}{6}(3a^2+h^2).
\end{aligned}$$

Part (b) Show that the volume volume of the region in the sphere bounded between the given disks (i.e., the *spherical segment*) equals $\frac{\pi h}{6}(3a^2+3b^2+h^2)$



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Solution. Let C_a denote the spherical cap with base the indicated disk of radius a and C_b denote the spherical cap with base the indicated disk with radius b. Let d denote the height of C_a and c denote the height of C_b. Let us denote by S the spherical segment whose volume we seek. Then

$$\text{volume}(S)=\text{volume}(C_a)-\text{volume}(C_b)=\frac{\pi d}{6}(3a^2+d^2)-\frac{\pi c}{6}(3b^2+c^2).$$

We must show that $\frac{\pi h}{6}(3a^2+3b^2+h^2)=\frac{\pi d}{6}(3a^2+d^2)-\frac{\pi c}{6}(3b^2+c^2)$. As in part (a), we have $R^2=(R-d)^2+a^2$ and $R^2=(R-c)^2+b^2$. The first equation implies that $(*)$ $a^2=2Rd-d^2$ and the second equation implies that $(**)$ $b^2 = 2Rc - c^2$. We will use these equations below, along with $h = d - c$. We have:

$$\begin{aligned}
\frac{\pi h}{6}(3a^2+3b^2+h^2) &= \frac{\pi}{6}(d-c)(3a^2+3b^2+h^2) \\
&= \frac{\pi d}{6}(3a^2+3b^2+h^2) - \frac{\pi c}{6}(3a^2+3b^2+h^2) \\
&= \frac{\pi d}{6}(3a^2+3b^2+d^2-2dc+c^2) - \frac{\pi c}{6}(3a^2+3b^2+d^2-2cd+c^2) \\
&= \frac{\pi d}{6}(3a^2+d^2) + \frac{\pi d}{6}(3b^2-2dc+c^2) - \frac{\pi c}{6}(3b^2+c^2) - \frac{\pi c}{6}(3a^2+d^2-2cd) \\
&= \text{volume}(C_a) - \text{volume}(C_b) + \frac{\pi d}{6}(3b^2-2dc+c^2) - \frac{\pi c}{6}(3a^2+d^2-2cd) \\
&= \text{volume}(S) + \frac{\pi d}{6}(3b^2-2dc+c^2) - \frac{\pi c}{6}(3a^2+d^2-2cd).
\end{aligned}$$

To finish we must show that

$$\frac{\pi d}{6}(3b^2-2dc+c^2) - \frac{\pi c}{6}(3a^2+d^2-2cd) = 0,$$

and for this, it suffices to show that

$$d(3b^2-2dc+c^2) - c(3a^2+d^2-2cd) = 0.$$

Thus, we must show that

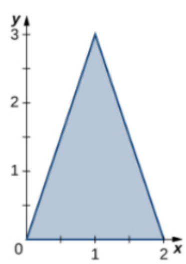
$$3db^2 - 2d^2c + dc^2 - 3ca^2 + 2c^2d - cd^2 = 0.$$

Simplifying, this is equivalent to showing that

$$(***) \qquad b^2d - a^2c + c^2d - cd^2 = 0.$$

Now, if we multiply (**) by d and (*) by c and subtract, we get $b^2d - a^2c = cd^2 - c^2d$. Substituting this into (***) gives 0, which is what we want.

5. OS Chapter 5: #389: This problem asks to find the area of the triangle R:



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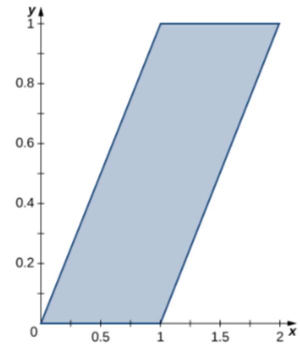
by finding a linear transformation T from the uv plane such that $T(0,0) = (0,0), T(1,0) = (2,0)$, and $T(0,1) = (1,3)$. This transformation will then take the triangle S in the uv-plane with vertices (0,0), (1,0), (0,1) to R.

Solution. From class we seen that we can take $T(u,v) = (2u+v, 3v)$. It is easy to check that $\text{Jac}(T) = -3$, so that $|\text{Jac}(T)| = 3$. Thus,

$$\begin{aligned}
\text{area}(R) &= \int\int_R dA \\
&= \int\int_S 3\ du\ dv \\
&= 3 \cdot \text{area}(S) \\
&= 3,
\end{aligned}$$

as expected.

5. OS Chapter 5: #391. Calculate $\int\int_R (y^2 - xy)\ dA$, for R



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for the given transformation.

Solution. The equations $u = y - x$ and $v = y$, can be rewritten as $x = v - u$ and $y = v$, which tells us our transformation should be $T(u, v) = (v - u, v)$. Substituting the vertices of R into the equations $u = y - x, v = y$ yields, vertices (0,0), (-1,0), (-1,1), (0,1) in the uv-plane, so that T transforms the rectangle $S = [-1, 0] \times [0, 1]$ in the uv-plane to R in the xy-plane. IT is easy to see that $\text{Jac}(T)| = 1$, so that

$$\begin{aligned}\int\int_R (y^2 - xy)\ dA &= \int_0^1 \int_0^1 vu\ dv\ du \\ &= \int_0^1 \frac{u}{2}\ du \\ &= \frac{1}{4}.\end{aligned}$$

5. OS Chapter 5: #431. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 16$, from $z = 1$ to $x + z = 2$.

Solution. We are finding the volume of the solid between the planes $z = 1$ and $z = 2 - x$, above the disk $D : 0 \leq x^2 + y^2 \leq 16$ in the xy-plane. Notice that if $x \geq 1$, then $2 - x \leq 1$ and if $x \leq 1$, then $1 \leq 2 - x$. Thus, the volume we seek is:

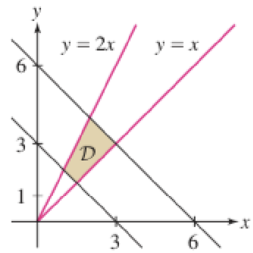
$$\int_{-4}^{1} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (2 - x) - 1\ dy\ dx + \int_{1}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1 - (2 - x)\ dy\ dx \qquad (\star)$$

For the first integral in $(\star)$ we have

$$\begin{aligned}\int_{-4}^{1} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (2 - x) - 1\ dy\ dx &= \int_{-4}^{1} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1 - x\ dy\ dx \\ &= \int_{-4}^{1} (1 - x)y\Big|_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}\ dx \\ &= 2\int_{-4}^{1} (1 - x)\sqrt{16 - 4x^2}\ dx \\ &\approx 71.78,\end{aligned}$$

the last single integral being worked numerically, though one could use the standard (complicated) formula for $\int \sqrt{1 - x^2}\ dx$ typically found on the inside cover of a calculus book. Similarly, second integral in $(\star)$ is approximately 21.51, so the required area is approximately 93.29.

6. Calculate $\int\int_D (x+y)\ dA$, for D



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using the transformation $G(u,v) = (\frac{u}{v+1}, \frac{uv}{v+1})$.

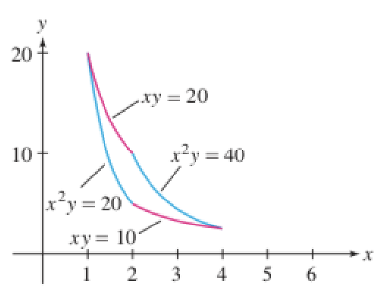
Solution. We need to find the region R in the uv-plan that $G(u,v)$ transforms to D. We use the equations of the lines bounding D. If $y = x$, then $\frac{u}{v+1} = \frac{uv}{v+1}$, from which we get $v = 1$. Similarly, the equation $y = 2x$ yields $v = 2$. The line in the xy plane containing (0,3) and (3,0) is $y = -x+3$. If we solve the corresponding equation $\frac{uv}{v+1} = -\frac{u}{v+1} + 1$ for u we get $u = 3$. Similarly, the lime through (0,6) and (6,0) in the xy plane gives rise to $u = 6$. Thus, the region R in the uv-plane is bounded by the lines $v = 1, v = 2, u = 3, u = 6$, so that $R = [3,6] \times [1,2]$. Calculating the Jacobian, we get

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{1}{v+1} & -\frac{u}{(v+1)^2} \\ \frac{v}{v+1} & \frac{u}{(v+1)^2} \end{pmatrix} = \frac{u}{(v+1)^3} + \frac{uv}{(v+1)^3} = \frac{u}{(v+1)^2}.$$

Since $3 \le u \le 6$, we have $|\frac{\partial(x,y)}{\partial(u,v)}| = \frac{u}{(v+1)^2}$. Thus,

$$\begin{aligned} \int\int_D (x+y)\ dA &= \int_3^6 \int_1^2 (\frac{u}{v+1} + \frac{uv}{v+1}) \cdot \frac{u}{(v+1)^2}\ dv\ du \\ &= \int_3^6 \int_0^1 \frac{u^2}{(v+1)^2}\ dv\ du \\ &= \int_3^6 u^2(-\frac{1}{v+1})_{v=1}^{v=2}\ du \\ &= \frac{1}{6}\int_3^6 u^2\ du \\ &= \frac{1}{6}(\frac{6^3}{3} - \frac{3^3}{3}) \\ &= \frac{21}{2}. \end{aligned}$$

7. Calculate $\int\int_D e^{xy}\ dA$, for D the region



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by using the inverse of the transformation $F(x,y) = (xy, x^2y)$.

Solution. To find $G(u,v)$, the inverse of $F(x,y)$, we use the equations $u = xy$ and $v = x^2y$ to solve for x and y in terms of u and v. These equations give $\frac{u}{x} = y = \frac{v}{x^2}$, and thus, $\frac{u}{x} = \frac{v}{x^2}$ yields $x = \frac{v}{u}$. Since $y = \frac{u}{x}$, we infer $y = \frac{u^2}{v}$. Thus, $G(u,v) = (\frac{v}{u}, \frac{u^2}{v})$. Note that when $xy = 10$ and $xy = 20$, then $u = 10$

and $u = 20$. This shows that $G(u,v)$ takes the lines $u = 10$ and $u = 20$ in the uv-plane to the hyperbolas $xy = 10$ and $xy = 20$ in the xy-plane. Similarly, $G(u,v)$ takes the lines $v = 20$ and $v = 40$ in the uv-plane to the graphs of $x^2y = 20$ and $x^2y = 40$ in the xy-plane. Now let's look at the four corners of the rectangle R in the uv-plane determined by the lines $u = 10, u = 20, v = 20, v = 40$. The lower left corner is (10, 20). $G(10,20) = (2,5)$ which is the lower left corner of the region D. $G(10,40) = (4,2.5)$ which is the lower right corner of D. Similarly, $G(u,v)$ takes the other two corners of R to the remaining corners of D, so it follows that G transforms R into D (by continuity of $G(u,v)$ and the fact that for the point (10, 30) in the interior of R, $G(10,30) = (3, \frac{10}{3})$ lies in the interior of D).

For the absolute value of the Jacobian of $G(u,v)$ we have

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ \frac{2u}{v} & -\frac{u^2}{v^2} \end{pmatrix}\right| = \left| -\frac{1}{v}\right| = \frac{1}{v}.$$

Thus,

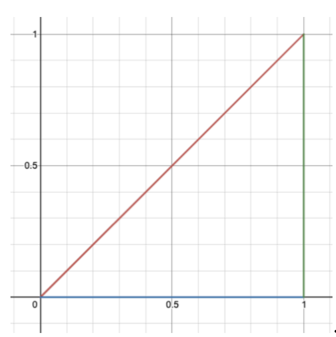
$$\begin{aligned} \int\int_D e^{xy}\ dA &= \int_{20}^{40}\int_{10}^{20} e^u \cdot \frac{1}{v}\ du\ dv \\ &= \int_{20}^{40} (e^{20} - e^{10}) \cdot \frac{1}{v}\ dv \\ &= (e^{20} - e^{10}) \int_{20}^{40} \frac{1}{v}\ dv \\ &= (e^{20} - e^{10}) \cdot (\ln(40) - \ln(20)) = (e^{20} - e^{10}) \cdot \ln(2). \end{aligned}$$

8. $\int\int_D \sqrt{x+y}(x-y)^2\ dA$, where D is the region bounded by the lines $x = 0, y = 0, x + y = 1$.

Solution. Because the integrand has no obvious ant-derivative with respect to either variable, we try to simplify it with a change of variables. If we choose u and v so that $u = x + y$ and $v = x - y$, then integrand then becomes $\sqrt{u}v^2$, which we can anti-differentiate. We can solve the system of equations $u = x + y$ and $v = x - y$ for x and y in terms of u and v and this will give the required change of variables. Upon doing so, we have $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Call this transformation $G(u,v)$. From this, it follows that

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2},$$

from which we get $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$. We now have to see what region in the uv-plane gets transformed to the region D in the xy plane, which is the triangle below:



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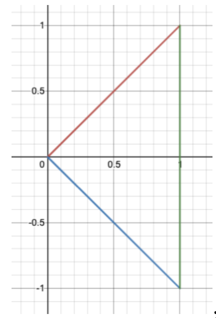
One edge of the triangle D is $x + y = 1$. In terms of u and v, this equation becomes $u = 1$. Thus, $G(u,v)$ transforms the line $u = 1$ in the uv plane to the line $x + y = 1$ in the xy-plane. Similarly, the equation $x = 0$ in terms of u and v becomes $u = y$, $v = -y$, so that $v = -u$, while the equation $y = 0$ yields $u = x, v = x$, so that $v = u$. Thus, if we let D_0 be the region in the uv-plane bounded by the lines $u = 1$, $v = -u$, and $v = u$,



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we see that $G(D_0) = D$. Thus,

$$\begin{aligned}\int\int_D \sqrt{x+y}(x-y)^2\ dA &= \int\int_{D_0} \sqrt{u}v^2\ \frac{1}{2}\ dA \\ &= \frac{1}{2}\int_0^1 \int_{-u}^{u} \sqrt{u}v^2\ dv\ du \\ &= \frac{1}{2}\int_0^1 \sqrt{u}(\frac{v^3}{3})_{v=-u}^{v=u}\ du \\ &= \frac{1}{6}\int_0^1 2u^{\frac{7}{2}}\ du \\ &= \frac{1}{3}\cdot\frac{2}{9}(u^{\frac{9}{2}})\Big|_0^1 \\ &= \frac{2}{27}.\end{aligned}$$

9. $\int\int_D \frac{1}{(x^2+y^2)^{\frac{3}{4}}}\ dA$, where D is the disk centered at the origin in $\mathbb{R}^2$ with radius R.

Solution. This is an improper double integral, as $f(x,y)$ is unbounded on D (since $\lim_{(x,y)\to(0,0)} f(x,y)$ tends to infinity). Let D_ϵ denote the region $\epsilon^2 \le x^2+y^2 \le R^2$, and we consider $\lim_{\epsilon\to 0}\int\int_{D_\epsilon} f(x,y)\ dA$. If this limit exists, it equals $\int\int_D \frac{1}{(x^2+y^2)^{\frac{3}{4}}}\ dA$. We have

$$\begin{aligned}\lim_{\epsilon\to 0}\int\int_{D_\epsilon} f(x,y)\ dA &= \lim_{\epsilon\to 0}\int\int_{D_\epsilon} \frac{1}{(x^2+y^2)^{\frac{3}{4}}}\ dA \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^R \frac{1}{(r^2)^{\frac{3}{4}}}\ r\ dr\ d\theta \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^R \frac{1}{r^{\frac{3}{2}}}\ r\ dr\ d\theta \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^R r^{-\frac{1}{2}}\ dr\ d\theta \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi} 2\sqrt{r}\Big|_\epsilon^R\ d\theta \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi} 2(\sqrt{R}-\sqrt{\epsilon})\ d\theta \\ &= \lim_{\epsilon\to 0} 4\pi(\sqrt{R}-\sqrt{\epsilon}) \\ &= 4\pi\sqrt{R}.\end{aligned}$$

10. $\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\ dA.$

Solution. We may test convergence of the double integral by integrating over increasing disks D_R of radius R centered at the origin. If the limit exists as $R \to \infty$, it equals $\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\ dA$.

$$\begin{aligned}
\lim_{R\to\infty} \int\int_{D_R} e^{-(x^2+y^2)}\ dA &= \lim_{R\to\infty} \int_0^{2\pi} \int_0^R e^{-r^2}\ r\ dr\ d\theta \\
&= \lim_{R\to\infty} \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=R}\ d\theta \\
&= \lim_{R\to\infty} \int_0^{2\pi} -\frac{1}{2}e^{-R^2} + \frac{1}{2}\ d\theta \\
&= \lim_{R\to\infty} 2\pi(-\frac{1}{2}e^{-R^2} + \frac{1}{2}) \\
&= 2\pi(0 + \frac{1}{2}) \\
&= \pi.
\end{aligned}$$

11. $\int\int_D \frac{1}{x^2y^2}\ dA$, where D is the set of points in $\mathbb{R}^2$ satisfying $2 \le x \le \infty$ and $2 \le y \le \infty$.

Solution. We may test convergence of the double integral by integrating increasing rectangles (or squares) whose lower left corner is (2,2). Let D_a denote the square $[2, a] \times [2, a]$ with $2 \le a < \infty$. If the limit exists as $a \to \infty$, it equals $\int\int_D \frac{1}{x^2y^2}\ dA$.

$$\begin{aligned}
\lim_{a\to\infty} \int\int_{D_a} \frac{1}{x^2y^2}\ dA &= \lim_{a\to\infty} \int_2^a \int_2^a \frac{1}{x^2y^2}\ dy\ dx \\
&= \lim_{a\to\infty} \int_2^a -\frac{1}{x^2y}\Big|_{y=2}^{y=a}\ dx \\
&= \lim_{a\to\infty} \int_2^a -\frac{1}{ax^2} + \frac{1}{2x^2}\ dx \\
&= \lim_{a\to\infty} (\frac{1}{ax} - \frac{1}{2x})\Big|_{x=2}^{x=a} \\
&= \lim_{a\to\infty} \{(\frac{1}{a^2} - \frac{1}{2a}) - (\frac{1}{2a} - \frac{1}{4})\} \\
&= \frac{1}{4}
\end{aligned}$$

12. Compare your answer in problem 11 with $(\int_2^\infty \frac{1}{x^2}\ dx)^2$. Can you explain the relation between these two answers?

Solution. A calculation similar, though easier, than the one above shows that $\lim_{a\to\infty} \int_2^a \frac{1}{x^2}\ dx = \frac{1}{2}$. The answer in problem 12 is the square of the answer in problem 11, since

$$\begin{aligned}
\int_2^a \int_2^a \frac{1}{x^2y^2}\ dy\ dx &= \int_2^a \{\int_2^a \frac{1}{x^2y^2}\ dy\}\ dx \\
&= \int_2^a \frac{1}{x^2}\{\int_2^a \frac{1}{y^2}\ dy\}\ dx \\
&= \{\int_2^a \frac{1}{y^2}\ dy\} \int_2^a \frac{1}{x^2}\ dx \\
&= \{\int_2^a \frac{1}{y^2}\ dy\}^2,
\end{aligned}$$

and the limit of a square is the square of the limits, assuming both limits exist.